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Rate of convergence of the Pólya algorithm from polyhedral sets

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Abstract

In this paper we consider a problem of best approximation in ℓ_p, $1 < p \leqslant \infty$. Let h_p denote the best p-approximation of $h \in \mathbb{R}^n$ from a closed, convex set K of $\mathbb{R}^n$, $1 < p < \infty$, $h \notin K$, and let h^*_∞ be the strict uniform approximation of h from K. We prove that if K satisfies locally a geometrical property, fulfilled by any polyhedral set of $\mathbb{R}^n$, then $\lim\sup_{p\to\infty} p\,\|h_p - h^*_\infty\| < \infty$.

Keywords: Strict uniform approximation; Best discrete ℓ_p-approximation; Rate of convergence; Pólya algorithm; Polyhedral sets

1. Introduction

For $x = (x(1), x(2), \ldots, x(n)) \in \mathbb{R}^n$ and $1 \leqslant p \leqslant \infty$, the ℓ_p-norms are defined by

$$\|x\|_p = \left(\sum_{j=1}^{n} |x(j)|^p\right)^{1/p}, \qquad 1 \leqslant p < \infty,$$
$$\|x\| := \|x\|_\infty = \max_{1\leqslant j\leqslant n} |x(j)|, \qquad p = \infty.$$

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Let $K \neq \emptyset$ be a subset of $\mathbb{R}^n$. For a fixed $h \in \mathbb{R}^n \setminus K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best p-approximation of h from K if

$$\|h_p - h\|_p \leqslant \|f - h\|_p \quad \text{for all } f \in K.$$

Throughout this paper we will assume that K is a closed, convex set of $\mathbb{R}^n$. We also suppose $0 \notin K$ and $h = 0$. This involves no loss of generality since all relevant properties are translation invariant. In this context, the existence of h_p is a well-known result. Moreover, for $1 < p < \infty$, h_p is unique and characterized by (see for instance [12])

$$\sum_{j=1}^{n} (h_p(j) - f(j))|h_p(j)|^{p-1}\operatorname{sgn}(h_p(j)) \leqslant 0 \quad \text{for all } f \in K.$$

This condition can be written

$$\langle h_p - f, \varphi_p \rangle \leqslant 0 \quad \text{for all } f \in K, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$ and $\varphi_p \in \mathbb{R}^n$ is given by $\varphi_p(j) := |h_p(j)|^{p-1} \operatorname{sgn}(h_p(j))$, $1 \leqslant j \leqslant n$.

If $p = \infty$ we will also say that h_∞ is a best uniform approximation of 0 from K. A best uniform approximation may not be unique.

It is also known [1,4,6] that if K is an affine subspace, then

$$\lim_{p\to\infty} h_p = h_\infty^*, \tag{2}$$

where h_∞^* is a particular best uniform approximation of 0 from K, called the *strict* uniform approximation [6,10] and whose definition is also valid in any closed, convex set K. The strict uniform approximation is determined by the next property. Let H denote the set of the best uniform approximations of 0 from K. For every $h_\infty \in H$ we consider the vector $\tau(h_\infty)$ whose coordinates are given by $|h_\infty(j)|$, $1 \leqslant j \leqslant n$, arranged in decreasing order. Then h_∞^* is the only element in H which has $\tau(h_\infty^*)$ minimal in the lexicographic ordering.

In the literature the convergence (2) is called the *Pólya Algorithm* [9]. In [3,8] it is proved that if K is an affine subspace then a stronger result holds, namely that

$$\limsup_{p\to\infty} p\,\|h_p - h_\infty^*\| < \infty. \tag{3}$$

Moreover, in [8,10] it is deduced that there are constants $M_1, M_2 > 0$ and $0 \leqslant a \leqslant 1$, depending on K, such that

$$M_1\, a^p \leqslant p\,\|h_p - h_\infty^*\| \leqslant M_2\, a^p \quad \text{for all } p > 1.$$

The next example (see [2,6]) shows that if K is not an affine subspace, then h_p does not converge necessarily to the strict uniform approximation. On the other hand, in [4,6,7] we can find a sufficient condition on K under which (2) holds. In particular, if K is a polyhedral set, i.e., a finite intersection of closed half-spaces, then K satisfies this condition. Furthermore, in this case it is proved in [5] that (3) holds.

Example 1. Let $K \subset \mathbb{R}^3$ be the convex hull of

$$\{(x, y, z) : y = 1 + (x - 1)^2, \quad 0 \leqslant x \leqslant 1,\ z = 1\} \cup \{0, 0, 0\}.$$

In this case, $h^*_\infty = (1, 1, 0)$ and it is not difficult to prove that $\lim_{p\to\infty} h_p = (1, 1, 1)$.

Henceforth, without loss of generality, we will assume $\|h^*_\infty\| = 1$. Let $1 = d_1 > d_2 > \cdots > d_s \geqslant 0$ denote all the different values of $|h^*_\infty(j)|$, $1 \leqslant j \leqslant n$, and let $\{J_r\}_{r=1}^s$ be the partition of $J := \{1, 2, \dots, n\}$ defined by $J_r = \{j \in J : |h^*_\infty(j)| = d_r\}$.

Note that if h_∞ is any best uniform approximation of 0 from K, then $h_\infty(j) = h^*_\infty(j)$ for all $j \in J_1$; otherwise, $(h_\infty + h^*_\infty)/2$ should be a best uniform approximation of 0 from K that contradicts the definition of the strict uniform approximation. For all $p > 1$,

$$\|h^*_\infty\| \leqslant \|h_p\| \leqslant \|h_p\|_p \leqslant \|h^*_\infty\|_p \leqslant n^{1/p} \|h^*_\infty\|.$$

So the set $\{\|h_p\|\}_{p=1}^\infty$ is bounded, and $\lim_{p\to\infty} \|h_p\| = \|h^*_\infty\|$. It follows that the limit as $p \to \infty$ of any convergent subsequence of $\{h_p\}$ is necessarily a best uniform approximation of 0 from K. Then observe that, in particular, (2) is valid whenever h^*_∞ is the unique best uniform approximation of 0 from K. In general, since $h_\infty(j) = h^*_\infty(j)$ for all $j \in J_1$, we deduce that

$$\lim_{p\to\infty} h_p(j) = h^*_\infty(j) \quad \text{for all } j \in J_1.$$

Also, it is easy to prove that the function $F : (1, +\infty) \to \mathbb{R}^n$, given by $F(p) = h_p$, is continuous.

The next example is especially interesting because it presents a situation where h_p does not converge to any point as $p \to \infty$.

Example 2. Consider the curves C_1, C_2 in $\mathbb{R}^3$ given by

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 : y = 1 + (x - 1)^2, \quad 0 \leqslant x \leqslant 1,\ z = 0\},$$
$$C_2 = \{(x, y, z) \in \mathbb{R}^3 : y = 1 + (x - 1)^2, \quad 0 \leqslant x \leqslant 1,\ z = 1\}.$$

For every integer $k \geqslant 2$ take $P_k = \left(1 - \frac{2}{2k-1}, 1 + \frac{4}{(2k-1)^2}, 0\right) \in C_1$, $Q_k = \left(1 - \frac{1}{k}, 1 + \frac{1}{k^2}, 1\right) \in C_2$. Let T_k be the convex hull of the points P_k, Q_k, P_{k+1} and let T'_k be the convex hull of the points Q_k, P_{k+1} and Q_{k+1}. Finally, let K denote the closed convex hull of $\bigcup_{k\geqslant 2} (T_k \cup T'_k)$. Observe that the segment $L := \{(1, 1, t) : 0 \leqslant t \leqslant 1\}$ is in K. Moreover, since $h(2) \geqslant 1$ for all $h \in K$, it is easy to prove that the set of the best uniform approximations of 0 from K is precisely L, and so $h^*_\infty = (1, 1, 0)$.

If we write $h_p = (x_p, y_p, z_p)$, $p > 1$, then $(x_p, y_p) \to (1, 1)$ as $p \to \infty$. As $(1, 1, 0) \in K$, we have $x_p^p + y_p^p + z_p^p \leqslant 2$ for all $p > 1$, and hence $h_p \neq (1, 1, t)$, with $0 < t \leqslant 1$. Applying (1) we see easily that $h_p \neq (1, 1, 0)$. Thus for all $p > 1$ we get $x_p^p + y_p^p + z_p^p < 2$ and it is deduced that $x_p < 1$, $y_p > 1$ and $0 \leqslant z_p < 1$.

We have $x_p^p \to 0$ and $z_p^p \to 0$ as $p \to \infty$. Indeed, otherwise we can take a subsequence $p_k \to \infty$ such that $(x_{p_k}^{p_k}, y_{p_k}^{p_k}, z_{p_k}^{p_k}) \to (\alpha, \beta, \gamma)$ as $k \to \infty$, with $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $\alpha + \gamma > 0$. Observe that $\beta \geqslant 1$ since $y_{p_k} > 1$ for all k. Using a subsequence if necessary, we can suppose

$(x_{p_k}, y_{p_k}, z_{p_k}) \to (1, 1, z_0)$ as $k \to \infty$, with a fixed $z_0 \in [0, 1]$. Applying (1) we obtain

$$(x_{p_k} - h(1))x_{p_k}^{p_k-1} + (y_{p_k} - h(2))y_{p_k}^{p_k-1} + (z_{p_k} - h(3))z_{p_k}^{p_k-1} \leqslant 0 \quad \text{for all } h \in K,$$

and so, taking limits as $k \to \infty$,

$$\alpha(1 - h(1)) + \beta(1 - h(2)) + \gamma(z_0 - h(3)) \leqslant 0 \quad \text{for all } h \in K. \tag{4}$$

Note that $\gamma = 0$ if $0 \leqslant z_0 < 1$. Furthermore, if $z_0 = 1$ and $\gamma > 0$, we get a contradiction in (4) for $h = (1, 1, 0)$. So $\gamma = 0$ and (we are assuming) $\alpha > 0$. Now, applying (4) with $h = P_k \in K$, we get $\alpha - \frac{2\beta}{2k-1} \leqslant 0$ for every integer $k \geqslant 2$. Hence $\alpha \leqslant 0$, a contradiction. Thus $x_p^p \to 0$ as $p \to \infty$. In particular, we have proved that $p\,(1 - x_p) \to +\infty$ as $p \to \infty$.

Since $x_p \uparrow 1$ as $p \to \infty$, the continuity of the map $p \mapsto x_p$ implies that there exists $p_k \to \infty$ as $k \to \infty$ such that $x_{p_k} = 1 - 1/k$ for all k large enough (thus $p_k/k \to \infty$ as $k \to \infty$). Recall that $z_{p_k} < 1$ for every p_k. Furthermore, it is easy to see that $z_{p_k} \neq 0$ whenever $x_{p_k} = 1 - 1/k$. Thus for k large enough $h_{p_k} \in \text{int}(T_k)$, and so h_{p_k} coincides with the best p_k-approximation of 0 from the plane π_k, determined by the points P_k, Q_k and P_{k+1}, and whose equation is given by

$$8k^3x + k^2(4k^2 - 1)y + z = 4k^4 + 8k^3 - 5k^2.$$

Since h_{p_k} is the only point of contact of the surface $x^p + y^p + z^p = x_{p_k}^{p_k} + y_{p_k}^{p_k} + z_{p_k}^{p_k}$ with the plane π_k, for k sufficiently large there exists $\lambda_k \neq 0$ such that

$$(x_{p_k}^{p_k-1}, y_{p_k}^{p_k-1}, z_{p_k}^{p_k-1}) = \lambda_k(8k^3, k^2(4k^2 - 1), 1).$$

Hence, in particular, $z_{p_k}^{p_k-1} = x_{p_k}^{p_k-1}/8k^3$, and so $z_{p_k} = x_{p_k}/(8k^3)^{1/(p_k-1)}$. Since $p_k/k \to \infty$ and $x_{p_k} \to 1$, as $k \to \infty$, we obtain immediately $\lim_{k\to\infty} z_{p_k} = 1$, and hence $\lim_{k\to\infty} h_{p_k} = (1, 1, 1)$.

Similarly, there exists $p'_k \to \infty$ as $k \to \infty$ such that $x_{p'_k} = 1 - 2/(2k - 1)$ for all k large enough. For these k it is immediate to prove that $h_{p'_k} = P_k$, and so $\lim_{k\to\infty} h_{p'_k} = (1, 1, 0)$. Consequently, h_p does not converge as $p \to \infty$. Furthermore, as the map $p \mapsto z_p$ is continuous for $p \in (1, \infty)$, for each $z_0 \in [0, 1]$ there exists a subsequence $p''_k \to \infty$ such that $\lim_{k\to\infty} h_{p_k''} = (1, 1, z_0)$.

The main purpose of this paper is to give a condition on K so that (3) holds. The following example shows that, even in the case that $h_p \to h^*_\infty$, (3) may not be achieved.

Example 3. In $\mathbb{R}^2$, let K be the convex hull of the curve $C = \{(x, y) : y = 1 + (x - 1)^2,\ 0 \leqslant x \leqslant 1\}$. An easy computation shows that $h^*_\infty = (1, 1)$ is the unique best uniform approximation of 0 from K and that $h_p = (1 - \delta_p, 1 + \delta_p^2)$, with $\delta_p > 0$ for all $p > 1$ and $\delta_p \to 0$ as $p \to \infty$. Since

$$(1 - \delta_p)^p + (1 + \delta_p^2)^p < 1^p + 1^p = 2,$$

we deduce that $\limsup(1 + \delta_p^2)^p < \infty$ as $p \to \infty$. Furthermore, since h_p is the only point of contact of the ℓ_p-ball with the curve C, we have

$$(1 - \delta_p)^{p-1}/(1 + \delta_p^2)^{p-1} = 2\delta_p.$$

We conclude that $\lim_{p\to\infty}(1-\delta_p)^{p-1} = 0$, and so $\lim_{p\to\infty} e^{-\delta_p(p-1)} = 0$, whence $p\,\delta_p \to \infty$ as $p \to \infty$.

2. The property A_∞

For $h \in K$, we define

$$V_h(K) = \{v \in \mathbb{R}^n : \|v\| = 1 \text{ and } h + \lambda\, v \in K \text{ for some } \lambda > 0\},$$

$$\rho_h(v) = \max\{\lambda : 0 < \lambda \leqslant 1,\ h + \lambda\, v \in K\} \quad \text{for each } v \in V_h(K),$$

$$\text{and } A_h = A_h(K) = \inf\{\rho_h(v) : v \in V_h(K)\}.$$

Definition 2.1. We say that K satisfies property A_∞ if $A_{h_\infty} > 0$ for every best uniform approximation h_∞ of 0 from K.

If K is a closed half-space of $\mathbb{R}^n$ then it is easy to prove that $A_h > 0$ for each $h \in K$. Moreover, a standard argument shows that the property "$A_h > 0$ for every element h in K" is preserved under finite intersection of closed, convex sets. In particular, if K is a polyhedral set, then $A_h > 0$ for all h in K, and therefore K satisfies property A_∞.

Theorem 2.2. *Let K be a nonempty, closed convex set of $\mathbb{R}^n$, $0 \notin K$. Let h_p denote the best p-approximation of 0 from K, $1 < p < \infty$, and let h^*_∞ be the strict uniform approximation of 0 from K. If K satisfies property A_∞ then* $\limsup p\,\|h_p - h^*_\infty\| < \infty$ *as* $p \to \infty$.

Proof. If the theorem is false, then there exists a sequence $p_k \uparrow \infty$ such that $p_k\|h_{p_k} - h^*_\infty\| \to \infty$ as $k \to \infty$. Thus we will prove the theorem by showing that for any sequence $p_k \uparrow \infty$, $\liminf_{k\to\infty} p_k\|h_{p_k} - h^*_\infty\| < \infty$. So let $p_k \uparrow \infty$ as $k \to \infty$. If $h_{p_k} = h^*_\infty$ for infinitely many k, then the result follows. Hence, using a subsequence if necessary, we can suppose $h_{p_k} \neq h^*_\infty$ for all k, and moreover $\lim_{k\to\infty} u_k = u$, with $\|u\| = 1$, where

$$u_k := \frac{h_{p_k} - h^*_\infty}{\|h_{p_k} - h^*_\infty\|}.$$

We assert that there exists $j_0 \in J$ for which $u(j_0) \neq 0$ and

$$\liminf_{k\to\infty}\ p_k|h_{p_k}(j_0) - h^*_\infty(j_0)| < \infty.$$

This assertion proves the theorem. Indeed,

$$\begin{aligned}\liminf_{k\to\infty}\ p_k\,\|h_{p_k} - h^*_\infty\| &= \liminf_{k\to\infty}\ \frac{p_k|h_{p_k}(j_0) - h^*_\infty(j_0)|}{|u_k(j_0)|}\\ &= \frac{1}{|u(j_0)|}\ \liminf_{k\to\infty}\ p_k|h_{p_k}(j_0) - h^*_\infty(j_0)| < \infty.\end{aligned}$$

Therefore our aim is now to prove that assertion.

In the following claim it is significant that $A_{h^*_\infty} > 0$.

Claim. *Both $h^*_\infty + \lambda u_k$ and $h^*_\infty + \lambda u$ are in K for every $0 \leqslant \lambda \leqslant A_{h^*_\infty}$.*

Indeed, observe that $h^*_\infty + \|h_{p_k} - h^*_\infty\|\, u_k = h_{p_k} \in K$, with $\|h_{p_k} - h^*_\infty\| > 0$, $\|u_k\| = 1$. Hence, $h^*_\infty + A_{h^*_\infty}\, u_k \in K$ for any k, and so $h^*_\infty + A_{h^*_\infty}\, u \in K$. As K is convex, we conclude that $h^*_\infty + \lambda\, u_k \in K$ and $h^*_\infty + \lambda\, u \in K$ for every $0 \leqslant \lambda \leqslant A_{h^*_\infty}$. So the claim is proved.

By the definition of h_{p_k} we have

$$|h_{p_k}(j_0)|^{p_k} \leqslant \sum_{j\in J} \left|h_{p_k}(j)\right|^{p_k} < \sum_{j\in J} |h^*_\infty(j)|^{p_k} \leqslant n \quad \text{for all } j_0 \in J. \tag{5}$$

We now consider two exhaustive cases:

(a) $u(j_0)h^*_\infty(j_0) > 0$ for some $j_0 \in J_1$.

In this case, for large k, $\dfrac{h_{p_k}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)} > 0$ and hence (recall that we are assuming $|h^*_\infty(j_0)| = 1$)

$$\begin{aligned} |h_{p_k}(j_0)|^{p_k} &= \left|1 + \frac{h_{p_k}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)}\right|^{p_k} = \left(1 + \frac{h_{p_k}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)}\right)^{p_k} \\ &\geqslant 1 + p_k\, \frac{h_{p_k}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)} = 1 + p_k |h_{p_k}(j_0) - h^*_\infty(j_0)|. \end{aligned}$$

By (5) we deduce that $\liminf_{k\to\infty}\ p_k\, |h_{p_k}(j_0) - h^*_\infty(j_0)| < \infty$. Thus the theorem is proved in this case.

(b) $u(j)h^*_\infty(j) \leqslant 0$ for each $j \in J_1$.

In this case we now show that

$$u(j) = 0 \quad \text{for all } j \in J_1. \tag{6}$$

Indeed, the claim asserts that $h^*_\infty + \lambda u$, with $0 \leqslant \lambda \leqslant A_{h^*_\infty}$, is in K; if $u(j_0)h^*_\infty(j_0) < 0$ for some $j_0 \in J_1$ and $u(j)h^*_\infty(j) \leqslant 0$ for each $j \in J_1$, then $h^*_\infty + \lambda u$, with $\lambda > 0$ and small enough, is a best uniform approximation of 0 from K that contradicts the definition of the strict uniform approximation. So (6) holds and then we deduce that $\widehat{h}_\infty := h^*_\infty + \lambda_0\, u$ is a best uniform approximation of 0 from K for some small $\lambda_0 \in (0, A_{h^*_\infty}]$. Thus, $A_{\widehat{h}_\infty} > 0$.

Taking $f = h^*_\infty$ in (1) we obtain $\left\langle h_{p_k} - h^*_\infty, \varphi_{p_k}\right\rangle \leqslant 0$, and so

$$\langle u_k, \varphi_{p_k}\rangle \leqslant 0 \quad \text{for all } k.$$

We will prove that

$$\langle u, \varphi_{p_k}\rangle \leqslant 0 \quad \text{for } k \text{ sufficiently large.} \tag{7}$$

When $u_k = u$, (7) holds trivially. Assume now $u_k \neq u$. Defining

$$v_k = (u_k - u)/\|u_k - u\|,$$

we have

$$\widehat{h}_\infty + \lambda_0 \|u_k - u\|\, v_k = h^*_\infty + \lambda_0\, u + \lambda_0 \|u_k - u\| v_k = h^*_\infty + \lambda_0 u_k.$$

Then from the claim, $\widehat{h}_\infty + \lambda_0 \|u_k - u\|\, v_k$ is in K. Since $\widehat{h}_\infty$ is a best uniform approximation of 0 from K, it follows that $\widehat{h}_\infty + \mu\, v_k \in K$, where $\mu := A_{\widehat{h}_\infty}$. Applying (1) with

$f = \widehat{h}_\infty + \mu v_k$, we get

$$\begin{aligned}\left\langle h_{p_k} - \widehat{h}_\infty - \mu v_k, \varphi_{p_k}\right\rangle &= \langle h_{p_k} - h^*_\infty - \lambda_0 u - \mu v_k, \varphi_{p_k}\rangle \\ &= \left(\|h_{p_k} - h^*_\infty\| - \frac{\mu}{\|u_k - u\|}\right) \langle u_k, \varphi_{p_k}\rangle \\ &\quad + \left(\frac{\mu}{\|u_k - u\|} - \lambda_0\right) \langle u, \varphi_{p_k}\rangle \leqslant 0.\end{aligned}$$

Since $\langle u_k, \varphi_{p_k}\rangle \leqslant 0$, $\{\|h_{p_k} - h^*_\infty\|\}$ is bounded and $\|u_k - u\| \to 0$ as $k \to \infty$, if in addition k is sufficiently large we deduce that

$$\left(\|h_{p_k} - h^*_\infty\| - \frac{\mu}{\|u_k - u\|}\right) \langle u_k, \varphi_{p_k}\rangle \geqslant 0$$

and hence

$$\left(\frac{\mu}{\|u_k - u\|} - \lambda_0\right) \langle u, \varphi_{p_k}\rangle \leqslant 0.$$

So (7) holds. Accordingly, in what follows we will suppose that k is sufficiently large so that (7) is valid and also $\operatorname{sgn}(u_k(j)) = \operatorname{sgn}(u(j))$ for all $j \in J$ for which $u(j) \neq 0$.

Note that due to (6) we have $s \geqslant 2$. Let $r_0 := \min\{r \in \{2, 3, \ldots, s\} : u(j) \neq 0$ for some $j \in J_r\}$. Observe that $d_{r_0} > 0$. Otherwise, $r_0 = s$ and then for every $j \in J_s$ with $u(j) \neq 0$, $\operatorname{sgn}(h_{p_k}(j)) = \operatorname{sgn}(u(j))$. So $u(j)\operatorname{sgn}(h_{p_k}(j)) > 0$ and hence

$$\langle u, \varphi_{p_k}\rangle = \sum_{j \in J_s} u(j)|h_{p_k}(j)|^{p_k - 1}\operatorname{sgn}(h_{p_k}(j)) > 0,$$

which contradicts (7). Then

$$\begin{aligned}\frac{1}{d_{r_0}^{p_k - 1}}\langle u, \varphi_{p_k}\rangle &= \sum_{j \in J} u(j)\left|\frac{h_{p_k}(j)}{d_{r_0}}\right|^{p_k - 1} \operatorname{sgn}(h_{p_k}(j)) \\ &= \sum_{r = r_0}^{s}\sum_{j \in J_r} u(j)\left|\frac{h_{p_k}(j)}{d_{r_0}}\right|^{p_k - 1} \operatorname{sgn}(h_{p_k}(j)) \leqslant 0,\end{aligned} \tag{8}$$

where the inequality is due to (7). Whenever $u(j)\operatorname{sgn}(h_{p_k}(j)) < 0$ for some $j \in J_r$ with $r \geqslant r_0$, we obtain $\operatorname{sgn}\left(h_{p_k}(j) - h^*_\infty(j)\right) \neq \operatorname{sgn}(h_{p_k}(j))$, and so $\left|\frac{h_{p_k}(j)}{h^*_\infty(j)}\right| = \left|\frac{h_{p_k}(j)}{d_r}\right| < 1$. Since $d_r \leqslant d_{r_0}$ if $r \geqslant r_0$, we also have

$$\left|\frac{h_{p_k}(j)}{d_{r_0}}\right| \leqslant \left|\frac{h_{p_k}(j)}{d_r}\right| < 1.$$

Then (8) implies

$$\liminf_{k \to \infty}\left|\frac{h_{p_k}(j)}{d_{r_0}}\right| < \infty \qquad \text{for every } j \in J_r,\ r \geqslant r_0. \tag{9}$$

If $u(j)h^*_\infty(j) \leqslant 0$ for all $j \in J_{r_0}$, then $u(j_1)h^*_\infty(j_1) < 0$ for some $j_1 \in J_{r_0}$. Thus, using the claim, we deduce that for $\lambda > 0$ and small enough $h^*_\infty + \lambda u$ is a best uniform approximation of 0 from K that in addition contradicts the definition of the strict uniform approximation. Therefore there exists $j_0 \in J_{r_0}$ such that $u(j_0)h^*_\infty(j_0) > 0$. Then $\dfrac{h_{p_k}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)} > 0$. Using (9) we get

$$\liminf_{k\to\infty} \left|\frac{h_{p_k}(j_0)}{d_{r_0}}\right|^{p_k-1} = \liminf_{k\to\infty} \left(1 + \frac{h_{p_k}(j_0) - h^*_\infty(j_0)}{h^*_\infty(j_0)}\right)^{p_k-1} < \infty.$$

Finally, applying the same procedure as in case (a), we obtain

$$\liminf_{k\to\infty} p_k\, |h_{p_k}(j_0) - h^*_\infty(j_0)| < \infty. \qquad \square$$

From Theorem 2.2 the following result is immediately deduced.

Corollary 2.3. *Let K be a nonempty closed, convex set of* $\mathbb{R}^n$, $0 \notin K$. *Let* h_p *denote the best p-approximation of* 0 *from K*, $1 < p < \infty$, *and let* h^*_∞ *be the strict uniform approximation of* 0 *from K. If K satisfies property* A_∞ *then* $\lim_{p\to\infty} h_p = h^*_\infty$.

The following example shows that property A_∞ is not necessary for (3) to hold.

Example 4. For $\alpha > 0$ and $x \leqslant 1/2$ we consider the function

$$f_\alpha(x) = 1 + \exp\left[-(\tfrac{1}{2} - x)^{-\alpha}\right], \qquad f_\alpha\left(\tfrac{1}{2}\right) = 1,$$

which is convex for $t_\alpha \leqslant x \leqslant \frac{1}{2}$, where $t_\alpha := \frac{1}{2} - \left(\frac{\alpha}{\alpha+1}\right)^{1/\alpha}$.

Let K_α be the convex hull of the curve

$$C_\alpha = \left\{(x, y) \in \mathbb{R}^2 : y = f_\alpha(x), \quad t_\alpha \leqslant x \leqslant \tfrac{1}{2}\right\}.$$

In this example, $h^*_\infty = \left(\frac{1}{2}, 1\right)$ is the only best uniform approximation of 0 from K_α and $h_p = \left(\frac{1}{2} - \delta_p, 1 + \varepsilon_p\right)$, where $\varepsilon_p = \exp(-\delta_p^{-\alpha})$ and $\delta_p \downarrow 0$ as $p \to \infty$. Using a similar argument to that in Example 3, we obtain

$$\frac{\left(\frac{1}{2} - \delta_p\right)^{p-1}}{(1 + \varepsilon_p)^{p-1}} = \alpha\, \delta_p^{-\alpha-1} e^{-\delta_p^{-\alpha}}.$$

We have $\lim_{p\to\infty} \dfrac{\left(\frac{1}{2} - \delta_p\right)^{\frac{p-1}{p}}}{(1 + \varepsilon_p)^{\frac{p-1}{p}}} = 1/2$. Suppose $\liminf_{p\to\infty} p\, \delta_p^\alpha = 0$. Then it is easy to see that

$$\liminf_{p\to\infty} \alpha^{1/p}\, \delta_p^{(-\alpha-1)/p}\, e^{(-p\,\delta_p^\alpha)^{-1}} = \liminf_{p\to\infty} \frac{p^{(\alpha+1)/(\alpha p)}\, \alpha^{1/p}}{(p\delta_p^\alpha)^{(\alpha+1)/(\alpha p)}\, e^{(p\delta_p^\alpha)^{-1}}} = 0,$$

a contradiction. Suppose now $\limsup_{p\to\infty} p\,\delta_p^{\alpha}=\infty$. Then it is immediate to see that $\limsup_{p\to\infty} \alpha^{1/p}$ $\delta_p^{(-\alpha-1)/p}\, e^{(-p\,\delta_p^{\alpha})^{-1}} \geqslant 1$, a contradiction as well. We conclude that there are constants $M_1, M_2 > 0$ such that $M_1 \leqslant p\,\delta_p^{\alpha} \leqslant M_2$, and so $h_p \to \left(\frac{1}{2}, 1\right)$ as $p \to \infty$ at a rate exactly $1/p^{1/\alpha}$.

Remark 1. Note that Theorem 2.2 remains true if the condition $A_{h_\infty} > 0$ is satisfied by any best uniform approximation h_∞ in an arbitrary small neighborhood of h_∞^*. On the other hand, Example 1 shows that Theorem 2.2 is not true with the condition $A_{h_\infty} > 0$ for $h_\infty = h_\infty^*$ only. Observe that in this example, $A_{h_\infty^*} = 1$ but $A_{h_\infty} = 0$ for each best uniform approximation $h_\infty \neq h_\infty^*$.

Remark 2. In Example 2 we get the same approximation problem if K is replaced by the infinite intersection of closed half-spaces determined by the planes which contain the triangles T_k, T_k', $k \geqslant 2$, and the half-space $z \geqslant 0$. Thus (2), and hence (3) as well, is not true generally if K is the intersection of infinitely many closed half-spaces.

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